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# Martingale Structure of Skorohod Integral Processes

Giovanni PECCATI
Laboratoire de Statistique Théorique et Appliquée
Université Paris VI, France
email: giovanni.peccati@gmail.com

Michèle THIEULLEN
Laboratoire de Probabilités et Modèles Aléatoires
Universités Paris VI and Paris VII, France
email: mth@ccr.jussieu.fr

Ciprian A. TUDOR*
Laboratoire de Probabilités et Modèles Aléatoires
Universités Paris VI and Paris VII, France
email: tudor@ccr.jussieu.fr

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**Abstract**

Let the process $\{Y_t, t \in [0,1]\}$, have the form $Y_t = \delta\left(u\mathbf{1}_{[0,t]}\right)$, where $\delta$ stands for a Skorohod integral with respect to Brownian motion, and $u$ is a measurable process verifying some suitable regularity conditions. We use a recent result by Tudor (2004), to prove that $Y_t$ can be represented as the limit of linear combinations of processes that are products of forward and backward Brownian martingales. Such a result is a further step towards the connection between the theory of continuous-time (semi)martingales, and that of anticipating stochastic integration. We establish an explicit link between our results and the classic characterization, due to Duc and Nualart (1990), of the chaotic decomposition of Skorohod integral processes. We also explore the case of Skorohod integral processes that are time-reversed Brownian martingales, and provide an "anticipating" counterpart to the classic Optional Sampling Theorem for Itô stochastic integrals.

**Key words** – Malliavin calculus; Anticipating stochastic integration; Martingale theory; Stopping times.

**AMS 2000 classification** – 60G15; 60G40; 60G44; 60H05; 60H07

**Running title** –Martingale structure of integrals

## 1 Introduction

Let $\left(C_{[0,1]}, \mathcal{C}, \mathbb{P}\right) = (\Omega, \mathcal{F}, \mathbb{P})$ be the canonical space, where $\mathbb{P}$ is the law of a standard Brownian motion started from zero, and write $X = \{X_t : t \in [0,1]\}$ for the coordinate process. In this paper, we investigate some properties of *Skorohod integral processes* defined with respect to $X$, that is, measurable stochastic processes with the form

$$Y_t = \int_0^1 u_s \mathbf{1}_{[0,t]}(s)\, dX_s = \int_0^t u_s dX_s, \quad t \in [0,1], \tag{1}$$

*The new address of C.A. Tudor is: SAMOS/MATISSE Université de Pantheon - Sorbonne, Paris I, France.

where $\{u_s : s \in [0,1]\}$ is a suitably regular (and not necessarily adapted) process verifying

$$\mathbb{E}\left[\int_0^1 u_s^2 ds\right] < +\infty, \tag{2}$$

and the stochastic differential $dX$ has to be interpreted in the Skorohod sense (as defined in Skorohod (1975); see the discussion below, as well as Nualart and Pardoux (1988) or Nualart (1995, Chapters 1 and 3), for basic results concerning Skorohod integration). It is well known that if $u_s$ is adapted to the natural filtration of $X$ (noted $\{\mathcal{F}_s : s \in [0,1]\}$) and satisfies (2), then $Y_t$ is a stochastic integral process in the Itô sense (as defined e.g. in Revuz and Yor (1999)), and therefore $Y_t$ is a square-integrable $\mathcal{F}_t$ - martingale. In general, the martingale property of $Y_t$ fails when $u_s$ is not $\mathcal{F}_s$ - adapted, and $Y_t$ may have a path behavior that is very different from the ones of classical Itô stochastic integrals (see Barlow and Imkeller (1992), for examples of anticipating integral processes with very irregular trajectories). However, in Tudor (2004) it is proved that the class of Skorohod integral processes (when the integrand $u$ is sufficiently regular) coincides with the set of *Skorohod-Itô integrals*, i.e. processes admitting the representation

$$Y_t = \int_0^t \mathbb{E}\left[v_s \mid \mathcal{F}_{[s,t]^c}\right] dX_s, \quad t \in [0,1], \tag{3}$$

where $v$ is measurable and satisfies (2), $\mathcal{F}_{[s,t]^c} := \mathcal{F}_s \vee \sigma\{X_1 - X_r : r \geq t\}$, and for each fixed $t$ the stochastic integral is in the usual Itô sense (indeed, for fixed $t$, $X_s$ is a standard Brownian motion on $[0,t]$, with respect to the enlarged filtration $s \mapsto \mathcal{F}_{[s,t]^c}$).

The principal aim of this paper is to use representation (3), in order to provide an exhaustive characterization of Skorohod integral processes in terms of products of *forward* and *backward* Brownian martingales. In particular, we shall prove that a process $Y_t$ has the representation (1) (or, equivalently, (3)) if, and only if, $Y_t$ is the limit, in an appropriate norm, of linear combinations of stochastic processes of the type

$$Z_t = M_t \times N_t, \quad t \in [0,1],$$

where $M_t$ is a centered (forward) $\mathcal{F}_t$ - martingale, and $N_t$ is a $\mathcal{F}_{[0,t]^c}$ - backward martingale (that is, for any $0 \leq s < t \leq 1$, $N_t \in \mathcal{F}_{[0,t]^c}$ and $\mathbb{E}\left[N_s \mid \mathcal{F}_{[0,t]^c}\right] = N_t$ ). Such a representation accounts in particular for the well-known property of Skorohod integral processes (see e.g. Nualart (1995, Lemma 3.2.1):

$$\mathbb{E}\left[Y_t - Y_s \mid \mathcal{F}_{[s,t]^c}\right] = 0 \text{ for every } s < t, \tag{4}$$

playing in the anticipating calculus a somewhat analogous role as the martingale property in the Itô's calculus. We will see, in the subsequent discussion, that our characterization of processes such as $Y_t$ complements some classic results contained in Duc and Nualart (1990), where the authors study the multiple Wiener integral expansion of Skorohod integral processes.

The paper is organized as follows. In Section 2, we introduce some notation and discuss preliminary issues concerning the Malliavin calculus; in Section 3, the main results of the paper are stated and proved; in Section 4, we establish an explicit link between our results and those contained in Duc and Nualart (1990); in Section 5, we concentrate on a special class of Skorohod integral processes, whose elements can be represented as *time-reversed Brownian martingales*, and we state sufficient conditions to have that such processes are semimartingales in their own filtration; eventually, Section 6 discusses some relations between processes such as (1) and stopping times.

## 2 Notation and preliminaries

Let $L^2([0,1], dx) = L^2([0,1])$ be the Hilbert space of square integrable functions on $[0,1]$. In what follows, the notation

$$X = \left\{X(f) : f \in L^2([0,1])\right\}$$

will indicate an *isonormal Gaussian process* on $L^2([0,1])$, that is, $X$ is a centered Gaussian family indexed by the elements of $L^2([0,1])$, defined on some (complete) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that

$\mathbb{E}\left[X\left(f\right)X\left(g\right)\right]=\int_0^1 f\left(x\right)g\left(x\right)dx$ for every $f,g\in L^2\left([0,1]\right)$. We also introduce the standard Brownian motion $X_t=X\left(\mathbf{1}_{[0,t]}\right)$, $t\in[0,1]$, and note $L^2\left(\mathbb{P}\right)$ the space of square integrable functionals of $X$. The usual notation of Malliavin calculus is adopted throughout the sequel (see Nualart (1990)): for instance, $D$ and $\delta$ denote the (Malliavin) derivative operator and the Skorohod integral with respect to the Wiener process $X$. For $k\geq 1$ and $p\geq 2$, $\mathbb{D}^{k,p}$ denotes the space of $k$ times differentiable functionals of $X$, endowed with the norm $\left\|\cdot\right\|_{k,p}$, whereas $\mathbb{L}^{k,p}=L^p\left([0,1];\mathbb{D}^{k,p}\right)$. Note that $\mathbb{L}^{k,p}\subset Dom(\delta)$, the domain of $\delta$. Now take a Borel subset $A$ of $[0,1]$, and denote by $\mathcal{F}_A$ the $\sigma$-field generated by random variables with the form $X\left(f\right)$, where $f\in L^2\left([0,1]\right)$ is such that its support is contained in $A$. We recall that if $F\in\mathcal{F}_A$ and $F\in\mathbb{D}^{1,2}$, then

$$D_tF\left(\omega\right)=0 \text{ on } A^c\times\Omega. \tag{5}$$

We will also need the following integration by parts formula:

$$\delta(Fu)=F\delta(u)-\int_{[0,1]}D_sFu_sds \tag{6}$$

p.s. - $\mathbb{P}$, whenever $u\in Dom(\delta)$ and $F\in\mathbb{D}^{1,2}$ are such that $\mathbb{E}(F^2\int_{[0,1]}u_s^2ds)<\infty$.

Eventually, let us introduce, for further reference, the following families of $\sigma$-fields:

$$\begin{aligned}\mathcal{F}_t &= \sigma\left\{X_h:h\leq t\right\},\quad t\in[0,1];\\ \mathcal{F}_{[s,t]^c} &= \sigma\left\{X_h:h\leq s\right\}\vee\sigma\left\{X_1-X_h:h\geq t\right\},\quad 0\leq s<t\leq 1,\end{aligned}$$

and observe that, to simplify the notation, we will write $\mathcal{F}_{[0,t]^c}=\mathcal{F}_{t^c}$, so that $\mathcal{F}_{[s,t]^c}=\mathcal{F}_{t^c}\vee\mathcal{F}_s$.

# 3 Skorohod integral processes and martingales

Let $L_0^2\left(\mathbb{P}\right)$ denote the space of zero mean square integrable functionals of $X$. We write $Y\in\mathbf{BF}$ to indicate that the measurable stochastic process $Y=\{Y_t:t\in[0,1]\}$ can be represented as a finite linear combination of processes with the form

$$Z_t=\mathbb{E}\left[H_1\mid\mathcal{F}_t\right]\times\mathbb{E}\left[H_2\mid\mathcal{F}_{t^c}\right]=M_t\times N_t,\quad t\in[0,1], \tag{7}$$

where $H_1\in L_0^2\left(\mathbb{P}\right)$ and $H_2\in L^2\left(\mathbb{P}\right)$. Note that $M$ in (7) is a forward (centered) Brownian martingale, whereas $N$ is a backward Brownian martingale. For every measurable process $G=\{G_t:t\in[0,1]\}$, we also introduce the notation

$$V\left(G\right)=\sup_{\pi}\mathbb{E}\left[\sum_{j=0}^{m-1}\left(G_{t_j}-G_{t_{j+1}}\right)^2\right], \tag{8}$$

where $\pi$ runs over all partitions of $[0,1]$ with the form $0=t_0<t_1<...<t_m=1$. The following result shows that $\mathbf{BF}$ is in some sense dense in the class of Skorohod integral processes.

**Theorem 1** *Let $u\in\mathbb{L}^{k,p}$, with $k\geq 3$ and $p>2$. Then, there exists a sequence of processes*

$$\left\{Z_t^{(r)}:t\in[0,1]\right\},\quad r\geq 1,$$

*with the following properties:*

**(i)** *for every $r$, $Z^{(r)}\in\mathbf{BF}$;*

**(ii)** *for every $r$, $Z_t^{(r)}=\int_0^t\mathbb{E}\left[v_\alpha^{(r)}\mid\mathcal{F}_{[\alpha,t]^c}\right]dX_\alpha$, $t\in[0,1]$, where $v^{(r)}\in\mathbb{L}^{k-2,p}$;*

**(iii)** *for every $r$, $V\left(Z^{(r)}\right)<+\infty$ and $\lim_{r\to\infty}V\left(\delta\left(u\mathbf{1}_{[0,\cdot]}\right)-Z^{(r)}\right)=0$.*

Note that points (i) and (iii) of Theorem 1 imply that $Z^{(r)}$ converges to $\delta\left(u\mathbf{1}_{[0,\cdot]}\right)$ uniformly in $L^2(\mathbb{P})$. This implies that the convergence takes also place in the sense of finite dimensional distributions. Before proving Theorem 1, we need to state two simple results.

**Lemma 2** *Fix $k \geq 1$ and $p \geq 2$. Let $A_1$ and $A_2$ be two disjoint subsets of $[0,1]$, and let $\mathcal{F}_{A_i}$, $i = 1,2$, be the $\sigma$-field generated by random variables of the form $X(h\mathbf{1}_{A_i})$, $h \in L^2([0,1])$. Suppose that $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$ and also $F \in \mathbb{D}^{k,p}$. Then, $F$ is the limit in $\mathbb{D}^{k,p}$ of linear combinations of smooth random variables of the type*

$$G = G_1 \times G_2, \tag{9}$$

*where, for $i = 1,2$, $G_i$ is smooth and $\mathcal{F}_{A_i}$ - measurable.*

**Proof.** By definition, every $F \in \mathbb{D}^{k,p}$ can be approximated in the space $\mathbb{D}^{k,p}$ by a sequence of smooth polynomial functionals of the type

$$P_m = p_{n_{(m)}}\left(X\left(h_1^{(m)}\right), ..., X\left(h_{n_{(m)}}^{(m)}\right)\right), \quad m \geq 1,$$

where, for every $m$, $n_{(m)} \geq 1$, $p_{n_{(m)}}$ is a polynomial in $n_{(m)}$ variables and, for $j = 1, ..., n_{(m)}$, $h_j^{(m)} \in L^2([0,1])$. It is also easily checked that $\mathbb{E}\left[P_m \mid \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\right] \in \mathbb{D}^{k,p}$ for every $m$ and, since $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$,

$$\mathbb{E}\left[P_m \mid \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\right] \to F$$

in $\mathbb{D}^{k,p}$. To conclude, it is sufficient to prove that every random variable of the kind

$$Z = \mathbb{E}\left[(X(h_1))^{k_1} \cdots (X(h_n))^{k_n} \mid \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\right]$$

where $h_j \in L^2([0,1])$ and $k_j \geq 1$, can be represented as a linear combination of random variables such as (9). To see this, write $A_3 = [0,1] \setminus (A_1 \cup A_2)$, and use twice the binomial formula to obtain

$$\begin{aligned} (X(h_j))^{k_j} &= \sum_{l=0}^{k_j} \binom{k_j}{l} (X(h_j\mathbf{1}_{A_1}))^{k_j-l} (X(h_j\mathbf{1}_{A_2\cup A_3}))^l \\ &= \sum_{l=0}^{k_j}\sum_{a=0}^{l} \binom{k_j}{l}\binom{l}{a} (X(h_j\mathbf{1}_{A_1}))^{k_j-l} (X(h_j\mathbf{1}_{A_2}))^{l-a} (X(h_j\mathbf{1}_{A_3}))^a, \end{aligned}$$

thus implying that the functional $(X(h_1))^{k_1} \cdots (X(h_n))^{k_n}$ is a linear combination of random variables of the type

$$H = \prod_{j=1}^{n} (X(h_j\mathbf{1}_{A_1}))^{\gamma_{1,j}} (X(h_j\mathbf{1}_{A_2}))^{\gamma_{2,j}} (X(h_j\mathbf{1}_{A_3}))^{\gamma_{3,j}},$$

where $\gamma_{i,j} \geq 0$, $j = 1, ..., n$, $i = 1,2,3$. To conclude, use independence to obtain

$$\mathbb{E}\left[H \mid \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\right] = \mathbb{E}\left[\prod_{j=1}^{n} (X(h_j\mathbf{1}_{A_3}))^{\gamma_{3,j}}\right] \times \prod_{j=1}^{n} (X(h_j\mathbf{1}_{A_1}))^{\gamma_{1,j}} \prod_{j=1}^{n} (X(h_j\mathbf{1}_{A_2}))^{\gamma_{2,j}},$$

and therefore the desired conclusion. ■

**Remark** – Suppose that $F = I_n^X(h)$, $n \geq 1$, where $I_n^X$ stands for a multiple Wiener integral of order $n$. Then, $F \in \mathbb{D}^{k,p}$ for every $k \geq 1$ and $p \geq 2$. Moreover, the isometric properties of multiple integrals imply that $F$ can be approximated in $\mathbb{D}^{k,2}$, and therefore in $\mathbb{D}^{k,p}$ for every $p \geq 2$, by linear combinations of random variables with the form $H_n(X(h))$, where $H_n$ is an Hermite polynomial of the $n$th order and $h$ is an element of $L^2([0,1])$. In particular, if $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$ as in the statement of Lemma 2, the arguments contained in the above proof entail that $F$ is the limit in $\mathbb{D}^{k,p}$ of linear combinations of random variables of the type $G = G_1 \times G_2$, where, for $i = 1, 2$, $G_i$ is a $\mathcal{F}_{A_i}$ - measurable polynomial functional of order $\gamma_i \geq 0$ such that $\gamma_1 + \gamma_2 \leq n$.

The proof of the following result is trivial, and it is therefore omitted.

**Lemma 3** *Fix $k \geq 1$ and $p \geq 2$, as well as a partition $0 = t_0 < t_1 < ... < t_n = 1$ of $[0,1]$. Then, for every finite collection $\{F_j : j = 1, ..., n\}$ of elements of $\mathbb{D}^{k,p}$, the process*

$$u_t = \sum_{j=0}^{n-1} F_j \mathbf{1}_{(t_j, t_{j+1})}(t)$$

*is an element $\mathbb{L}^{k,p}$. Moreover, if $F_j^m \underset{m\to+\infty}{\to} F_j$ in $\mathbb{D}^{k,p}$, then, as $m \to +\infty$, the sequence of processes*

$$u_t^m = \sum_{j=0}^{n-1} F_j^m \mathbf{1}_{(t_j, t_{j+1})}(t)$$

*converges to $u$ in $\mathbb{L}^{k,p}$.*

**Proof of Theorem 1.** It is well known (see e.g. Duc and Nualart (1990)) that the process $t \mapsto Y_t = \delta\left(u \mathbf{1}_{[0,t]}\right)$ is such that $V(Y) < +\infty$. Moreover, according to Proposition 1 in Tudor (2004), $Y$ admits the (unique) representation

$$Y_t = \int_0^t \mathbb{E}\left[v_\alpha \mid \mathcal{F}_{[\alpha,t]^c}\right] dX_\alpha, \quad t \in [0,1], \tag{10}$$

where $v \in \mathbb{L}^{k-2,p}$. Now, for every partition $\pi$ of the type $0 = t_0 < ... < t_n = 1$, we introduce the step process

$$v_t^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left( \int_{t_i}^{t_{i+1}} \mathbb{E}\left[v_s \mid \mathcal{F}_{[t_i, t_{i+1}]^c}\right] ds \right) \mathbf{1}_{(t_i, t_{i+1})}(t), \quad t \in [0,1], \tag{11}$$

and we recall that $v^\pi \in \mathbb{L}^{k-2,p}$, and that $v^\pi$ converges to $v$ in $\mathbb{L}^{k-2,p}$ whenever the mesh of $\pi$, noted $|\pi|$, converges to zero. Now define $Y_t^\pi = \int_0^t \mathbb{E}\left[v_\alpha^\pi \mid \mathcal{F}_{[\alpha,t]^c}\right] dX_\alpha$. From the calculations contained in Tudor (2004, Proposition 2), we deduce that

$$V(Y - Y^\pi) \leq \|v - v^\pi\|_{1,2}^2, \tag{12}$$

and therefore that $V(Y^\pi) < +\infty$ and $V(Y - Y^\pi)$ converges to zero, as $|\pi| \to 0$. Now fix a partition $\pi$, and note, for $i = 0, ..., n-1$,

$$F_i^\pi := \frac{1}{t_{i+1} - t_i} \left( \int_{t_i}^{t_{i+1}} \mathbb{E}\left[v_s \mid \mathcal{F}_{[t_i, t_{i+1}]^c}\right] ds \right) \in \mathcal{F}_{[t_i, t_{i+1}]^c}. \tag{13}$$

Since for every $i$ and every $s$ such that $t_i \leq s \leq t_{i+1}$ and $s < t$,

$$\mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[s,t]^c}\right] = \mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[s,t]^c \cap [t_i, t_{i+1}]^c}\right] = \mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[t_i, t_{i+1} \vee t]^c}\right],$$

we obtain, using the properties (6) and (5)

$$\begin{aligned} Y_t^\pi &= \sum_{i=0}^{n-1} \int_0^t \mathbf{1}_{[t_i, t_{i+1}]}(s) \mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[t_i, t_{i+1} \vee t]^c}\right] dX_s \\ &= \sum_{i=0}^{n-1} \mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[t_i, t_{i+1} \vee t]^c}\right] \left(X_{t \wedge t_{i+1}} - X_{t_i}\right) \mathbf{1}_{(t \geq t_i)} \\ &= \sum_{i=0}^{n-1} Z_t^{(\pi,i)}, \end{aligned}$$

where $Z_t^{(\pi,i)} = \mathbb{E}\left[F_i^\pi \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right]\left(X_{t\wedge t_{i+1}} - X_{t_i}\right)\mathbf{1}_{(t\geq t_i)}$. Now fix $i = 0, ..., n-1$. Since $F_i^\pi$ is $\mathcal{F}_{[t_i,t_{i+1}]^c}$-measurable and $F_i \in \mathbb{D}^{k-2,p}$, thanks to Lemma 2 in the special case $A_1 = (0, t_i)$ and $A_2 = (t_{i+1}, 1)$, the random variable $F_i^\pi$ is the limit in the space $\mathbb{D}^{k-2,p}$ of a sequence of random variables of the type

$$G_m^{(i,\pi)} = \sum_{k=1}^{M_m} G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)}, \quad m \geq 1, \tag{14}$$

where, for every $m$, $M_m \geq 1$ and, for every $k$, $G_{m,k}^{(i,\pi,1)}, G_{m,k}^{(i,\pi,2)}$ are smooth and such that $G_{m,k}^{(i,\pi,1)} \in \mathcal{F}_{t_i}$, and $G_{m,k}^{(i,\pi,2)} \in \mathcal{F}_{t_{i+1}^c}$. This implies, thanks to Lemma 3, that the process

$$v_t^{m,\pi} = \sum_{i=0}^{n-1} G_m^{(i,\pi)} \mathbf{1}_{(t_i,t_{i+1})}(t), \quad t \in [0,1],$$

converges to $v^\pi$ in $\mathbb{L}^{k-2,p}$, and therefore, due to an inequality similar to (12), for every $\pi$ the sequence of processes

$$\begin{aligned}
Y_t^{m,\pi} &= \sum_{i=0}^{n-1} \int_0^t \mathbf{1}_{[t_i,t_{i+1}]}(s)\, \mathbb{E}\left[G_m^{(i,\pi)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right] dX_s \\
&= \sum_{i=0}^{n-1} \mathbb{E}\left[G_m^{(i,\pi)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right]\left(X_{t\wedge t_{i+1}} - X_{t_i}\right)\mathbf{1}_{(t\geq t_i)} \\
&= \sum_{i=0}^{n-1} \sum_{k=1}^{M_m} \mathbb{E}\left[G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right]\left(X_{t\wedge t_{i+1}} - X_{t_i}\right)\mathbf{1}_{(t\geq t_i)} \\
&= \sum_{i=0}^{n-1} \sum_{k=1}^{M_m} U_t^{(m,k,\pi,i)}, \quad m \geq 1,
\end{aligned}$$

is such that $V\left(Y^{m,\pi}\right) < +\infty$ and $\lim_{m\to+\infty} V\left(Y^\pi - Y^{m,\pi}\right) = 0$. We shall now show that $U^{(m,k,\pi,i)} \in \mathbf{BF}$. As a matter of fact,

$$\begin{aligned}
U_t^{(m,k,\pi,i)} &= \mathbb{E}\left[G_{m,k}^{(i,\pi,1)} G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right]\left(X_{t\wedge t_{i+1}} - X_{t_i}\right)\mathbf{1}_{(t\geq t_i)} \\
&= \left[G_{m,k}^{(i,\pi,1)}\left(X_{t\wedge t_{i+1}} - X_{t_i}\right)\mathbf{1}_{(t\geq t_i)}\right] \times \mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right] \\
&= M_t \times N_t.
\end{aligned} \tag{15}$$

Eventually, observe that $M_t = \int_0^t H_s dX_s$ where $H_s = G_{m,k}^{(i,\pi,1)} \mathbf{1}_{(t_i,t_{i+1})}(s)$, and therefore, since $H_s$ is $\mathcal{F}_s$ - predictable, $M_t$ is a Brownian martingale such that $M_0 = 0$; on the other hand,

$$\begin{aligned}
N_t &= \mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right] = \mathbb{E}\left[\mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{t_{i+1}^c}\right] \mid \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{t_{i+1}^c}\right] \mid \mathcal{F}_{(t_{i+1}\vee t)^c}\right] = \mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{(t_{i+1}\vee t)^c}\right],
\end{aligned} \tag{16}$$

and also

$$\begin{aligned}
N_t &= \mathbb{E}\left[\mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{t_{i+1}^c}\right] \mid \mathcal{F}_{(t_{i+1}\vee t)^c}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[G_{m,k}^{(i,\pi,2)} \mid \mathcal{F}_{t_{i+1}^c}\right] \mid \mathcal{F}_{t^c}\right] \\
&= \mathbb{E}\left[N_0 \mid \mathcal{F}_{t^c}\right],
\end{aligned} \tag{17}$$

so that $N_t$ is a backward martingale such that $N_1 = \mathbb{E}\left[G_{m,k}^{(i,\pi,2)}\right]$. As a consequence, we obtain that $U^{(m,k,\pi,i)}$, and therefore $Y^{m,\pi}$, is an element of $\mathbf{BF}$. We have therefore shown that for every $r \geq 1$

there exists a partition $\pi(r)$ and a number $m(r,\pi(r))$ such that $V\left(Y - Y^{\pi(r)}\right) \leq 1/(4r)$ and also $V\left(Y^{\pi(r),m(r,\pi(r))} - Y^{\pi(r)}\right) \leq 1/(4r)$. To conclude, set $Z^{(r)} := Y^{\pi(r),m(r,\pi(r))}$ and observe that

$$V\left(Y - Z^{(r)}\right) \leq 2\left[V\left(Y - Y^{\pi(r)}\right) + V\left(Y^{\pi(r),m(r,\pi(r))} - Y^{\pi(r)}\right)\right] \leq \frac{1}{r}.$$

■

The next result contains a converse to Theorem 1.

**Theorem 4** *Let the sequence $Z^{(n)} \in \mathbf{BF}$, $n \geq 1$, be such that $V\left(Z^{(n)}\right) < +\infty$ and*

$$\lim_{n,m\to+\infty} V\left(Z^{(n)} - Z^{(m)}\right) = 0.$$

*Then, there exists a process $\{Y_t : t \in [0,1]\}$ such that*

**(i)** *$Y_t$ admits a Skorohod integral representation;*

**(ii)** *$V(Y) < +\infty$ and $\lim_{n\to+\infty} V\left(Z^{(n)} - Y\right) = 0$.*

**Proof.** We shall first prove point (ii). Consider the trivial partition $t_0 = 0$, $t_1 = 1$. Then, the assumptions in the statement (remember that $Z_0^{(n)} = 0$) imply that $Z_1^{(n)}$ is a Cauchy sequence in $L^2(\mathbb{P})$. Moreover, since for every $t \in (0,1)$,

$$\lim_{n,m\to+\infty} \mathbb{E}\left[\left(Z_t^{(n)} - Z_t^{(m)}\right)^2 + \left(Z_t^{(n)} - Z_t^{(m)} - \left(Z_1^{(n)} - Z_1^{(m)}\right)\right)^2\right] = 0,$$

we readily obtain that for every $t \in [0,1]$ there exists $Y_t \in L^2(\mathbb{P})$ such that $Y_0 = 0$ and also $Z_t^{(n)} \to Y_t$ in $L^2(\mathbb{P})$. Now fix $\varepsilon > 0$; it follows from the assumptions that there exists $N \geq 1$ such that for every $n, m > N$ and for every partition $0 = t_0 < ... < t_M = 1$

$$\mathbb{E}\left[\sum_{j=0}^{M-1}\left(\left(Z_{t_{j+1}}^{(n)} - Z_{t_{j+1}}^{(m)}\right) - \left(Z_{t_j}^{(n)} - Z_{t_j}^{(m)}\right)\right)^2\right] \leq \varepsilon,$$

and therefore, letting $m$ go to infinity, we obtain that for $n > N$

$$\sup_{\pi} \mathbb{E}\left[\sum_{j=0}^{M-1}\left(\left(Z_{t_{j+1}}^{(n)} - Y_{t_{j+1}}\right) - \left(Z_{t_j}^{(n)} - Y_{t_j}\right)\right)^2\right] = V\left(Z^{(n)} - Y\right) \leq \varepsilon,$$

that entails $\lim_{n\to+\infty} V\left(Z^{(n)} - Y\right) = 0$. To conclude the proof of (ii), observe that, for $n > N$ as before,

$$V(Y) \leq 2\left(V\left(Y - Z^{(n)}\right) + V\left(Z^{(n)}\right)\right) \leq 2\left(\varepsilon + V\left(Z^{(n)}\right)\right) < +\infty.$$

Thanks to Proposition 2.3. in Duc and Nualart (1990), to show point (i) it is now sufficient to prove that for any $s < t$

$$\mathbb{E}\left[Y_t - Y_s \mid \mathcal{F}_{[s,t]^c}\right] = 0,$$

which is easily proven by using $L^2$ convergence as well as the fact that for every process $Z_t$ as in (7) we have

$$\mathbb{E}\left[Z_t - Z_s \mid \mathcal{F}_{[s,t]^c}\right] = N_t \mathbb{E}\left[M_t \mid \mathcal{F}_{[s,t]^c}\right] - M_s \mathbb{E}\left[N_s \mid \mathcal{F}_{[s,t]^c}\right] = 0.$$

■

# 4 Representation of finite chaos Skorohod integral processes

We say that the process $Y = \{Y_t : t \in [0,1]\}$ is a *finite chaos Skorohod integral process of order* $N \geq 0$ (written: $Y \in \mathbf{FS}_N$) if $Y_t = \delta\left(u\mathbf{1}_{[0,t]}\right)$ for some Skorohod integrable process $u_\alpha(\omega) \in L^2([0,1] \times \Omega)$ such that, for each $\alpha \in [0,1]$, the random variable $u_\alpha$ belongs to $\oplus_{j=0,...,N} C_j$, where $C_j$ represents the $j$th Wiener chaos associated to $X$. Note that if $Y \in \mathbf{FS}_N$, then, for each $t$, $Y_t \in \oplus_{j=0,...,N+1} C_j$. We also define $\mathbf{FS} = \cup_{N \geq 0} \mathbf{FS}_N$. The aim of this paragraph is to discuss the relations between the results of the previous section, and the representation of the elements of the class $\mathbf{FS}$ introduced in Duc and Nualart (1990)). To this end, we shall need some further notation (note that our formalism is essentially analogous to the one contained in the first part of Duc and Nualart (1990)).

For every $M \geq 2$ and every $1 \leq m \leq M$, we write $\mathbf{j}_{(m)} \subset \{1, ..., M\}$ to indicate that the vector $\mathbf{j}_{(m)} = (j_1, ..., j_m)$ has integer-valued components such that $1 \leq j_1 < j_2 < ... < j_m \leq M$. Note that $\mathbf{j}_{(M)} = (1, ..., M)$. We set $\mathbf{j}_{(0)} = \varnothing$ by definition, and also, given $\mathbf{x}_M = (x_1, ..., x_M) \in [0,1]^M$ and $\mathbf{j}_{(m)} = (j_1, ..., j_m) \subset \{1, ..., M\}$,

$$\mathbf{x}_{\mathbf{j}_{(m)}} := (x_{j_1}, ..., x_{j_m}) \quad ; \quad \mathbf{x}_{\mathbf{j}_{(0)}} := 0.$$

We use the following notation: (a) for every permutation $\sigma^M = \{\sigma(1), ..., \sigma(M)\}$ of $\{1, ..., M\}$, we set

$$\Delta_M^{\sigma^M} := \left\{(x_1, ..., x_M) \in [0,1]^M : 0 < x_{\sigma(M)} < ... < x_{\sigma(1)} < 1\right\}$$

and also write

$$\Delta_M^{\sigma_0^M} := \Delta_M = \left\{(x_1, ..., x_M) \in [0,1]^M : 0 < x_M < ... < x_1 < 1\right\}$$

for the simplex contained in $[0,1]^M$; (b) for every $m = 0, ..., M$ and $\mathbf{j}_{(m)} \subset \{1, ..., M\}$,

$$\Delta_M^{\mathbf{j}_{(m)}} := \left\{(x_1, ..., x_M) \in (0,1)^M : \max_{i \in \mathbf{j}_{(m)}} (x_i) < \min_{l \in \{1,...,M\} \backslash \mathbf{j}_{(m)}} (x_l)\right\},$$

where $\max_{i \in \varnothing}(x_i) := 0$ and $\min_{l \in \varnothing}(x_l) := 1$; (c) for every $t \in [0,1]$ and every $\mathbf{j}_{(m)} \subset \{1, ..., M\}$,

$$\Delta_M^{\mathbf{j}_{(m)}}(t) := \left\{(x_1, ..., x_M) \in (0,1)^M : \max_{i \in \mathbf{j}_{(m)}} (x_i) < t < \min_{l \in \{1,...,M\} \backslash \mathbf{j}_{(m)}} (x_l)\right\};$$

(d) for every $t \in [0,1]$,

$$A_{M,m}(t) = \bigcup_{\mathbf{j}_{(m)} \subset \{1,...,M\}} \Delta_M^{\mathbf{j}_{(m)}}(t).$$

**Remark** – Note that $\Delta_M^{\mathbf{j}_{(0)}} = \Delta_M^{\mathbf{j}_{(M)}} = (0,1)^M$ and, in general, for every $m = 0, ..., M$ and every $\mathbf{j}_{(m)} \subset \{1, ..., M\}$

$$\Delta_M^{\mathbf{j}_{(m)}} = \bigcup_{t \in \mathbb{Q} \cap (0,1)} \Delta_M^{\mathbf{j}_{(m)}}(t).$$

We have also the following relations,

$$A_{M,M}(t) = \Delta_M^{\mathbf{j}_{(M)}}(t) = (0,t)^M \quad ; \quad A_{M,0}(t) = \Delta_M^{\mathbf{j}_{(0)}}(t) = (t,1)^M,$$

and moreover, if $t \in \{0,1\}$ and $0 < m < M$, then $A_{M,m}(t) = \varnothing$.

The following result corresponds to properties (**B1**)-(**B3**) in Duc and Nualart (1990).

**Proposition 5** *Fix $M \geq 2$ and $0 \leq m \leq M$, and let the previous notation prevail. Then, (i)*

$$\bigcup_{\mathbf{j}_{(m)} \subset \{1,...,M\}} \Delta_M^{\mathbf{j}_{(m)}} = [0,1]^M, \quad a.e.\text{-}Leb,$$

*where Leb stands for Lebesgue measure; (ii) if $\mathbf{i}_{(m)}, \mathbf{j}_{(m)} \subset \{1,...,M\}$, then $\Delta_M^{\mathbf{j}_{(m)}} \cap \Delta_M^{\mathbf{i}_{(m)}} \neq \varnothing$ if, and only if, $\mathbf{i}_{(m)} = \mathbf{j}_{(m)}$; (iii) for any $t \in [0,1]$, if $m \neq m'$ and $0 \leq m, m' \leq M$, then $A_{M,m}(t) \cap A_{M,m'}(t) = \varnothing$, and also*

$$\bigcup_{m=0,...,M} A_{M,m}(t) = [0,1]^M, \quad a.e.\text{-}Leb.$$

The next fact is a combination of Theorems 1.3 and 2.1 in Duc and Nualart (1990), and gives a univocal characterization of the chaos expansion of the elements of **FS**. Note that, in the following, we will write $L_s^2\left([0,1]^k\right)$, $k \geq 2$, to indicate the set of symmetric functions on $[0,1]^k$ that are square integrable with respect to Lebesgue measure. Moreover, for any $k \geq 2$ and $f \in L_s^2\left([0,1]^k\right)$, the symbol $I_k^X(f)$ will denote the standard multiple Wiener-Itô integral (of order $k$) of $f$ with respect to $X$ (see e.g. Nualart (1995, 1998) for definitions). We will also use the notation $L_s^2([0,1]) = L^2([0,1])$ and, for $f \in L^2([0,1])$, $I_1^X(f) = X(f)$.

**Theorem 6 (Duc and Nualart)** *Let the above notation prevail, and fix $N \geq 0$. Then, the process $Y = \{Y_t : t \in [0,1]\}$ is an element of $\mathbf{FS}_N$ if, and only if, there exists a (unique) collection of kernels $\{f_{l,q} : 1 \leq q \leq l \leq N+1\}$ such that $f_{l,q} \in L_s^2\left([0,1]^l\right)$ for every $1 \leq q \leq l \leq N+1$ and*

$$Y_t = \sum_{l=1}^{N+1} \sum_{q=1}^{l} I_l^X\left(f_{l,q} \mathbf{1}_{A_{l,q}(t)}\right), \quad t \in [0,1]. \tag{18}$$

*Moreover, if condition (18) is satisfied*

$$\sum_{l=1}^{N+1} l! \sum_{q=0}^{l-1} \|f_{l,q} - f_{l,q+1}\|^2 \leq V(Y) < +\infty, \tag{19}$$

*where $V(Y)$ is defined according to (8), and $f_{l,0} := 0$.*

The link between the objects introduced in this paragraph and those of the previous section is given by the following

**Lemma 7** *Fix $m, n \geq 0$, and for every $r \geq 1$ take a natural number $M_r \geq 1$, as well as two collections of kernels*

$$\left\{h_j^{(u,r)} : 1 \leq u \leq M_r; j = 1,...,m\right\} \quad ; \quad \left\{g_i^{(u,r)} : 1 \leq u \leq M_r; i = 1,...,n\right\},$$

*where $h_j^{(u,r)} \in L_s^2\left([0,1]^j\right)$ and $g_i^{(u,r)} \in L_s^2\left([0,1]^i\right)$ for every $i, j$, and a set of real numbers*

$$\left\{b^{(u,r)} : 1 \leq u \leq M_r\right\}.$$

*For every $t \in [0,1]$ and $r \geq 1$, we define*

$$Z_t^{(r)} := \sum_{u=1}^{M_r} Z_t^{(u,r)} = \sum_{u=1}^{M_r} \left(\sum_{j=1}^{m} I_j^X\left(h_j^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes j}\right)\right) \times \left(b^{(u,r)} + \sum_{i=1}^{n} I_i^X\left(g_i^{(u,r)} \mathbf{1}_{(t,1)}^{\otimes i}\right)\right). \tag{20}$$

*Then: (i) for every $r \geq 1$, $V\left(Z^{(r)}\right) < +\infty$; (ii) if*

$$\lim_{r,r'\uparrow+\infty} V\left(Z^{(r)} - Z^{(r')}\right) = 0,$$

*there exists a process $Y = \{Y_t : t \in [0,1]\}$ such that*

$$Y_0 = 0, \ \ V(Y) < +\infty \ \ \text{and} \ \ \lim_{r\uparrow+\infty} V\left(Z^{(r)} - Y\right) = 0, \tag{21}$$

*and moreover there exist a unique collection of kernels $f_{l,q} \in L_s^2\left([0,1]^l\right)$ such that, for every $t \in [0,1]$, $Y_t$ admits the representation*

$$Y_t = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1 \leq q \leq l\wedge m} I_l^X\left(\mathbf{1}_{A_{l,q}(t)} f_{l,q}\right), \quad t \in [0,1], \tag{22}$$

*where, for every $k \geq 1$, we adopt the notation $\sum_{k\leq q\leq 0} := 0$. In particular, $Y \in \mathbf{FS}_{n+m-1}$.*

**Proof.** If $m$ or $n$ is equal to zero, the statement can be proved by standard arguments. Now suppose $n, m \geq 1$, and fix $r \geq 1$ and $u = 1, ..., M_r$. The multiplication formula for multiple Wiener integrals yields

$$Z^{(u,r)} = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1 \leq q \leq l\wedge m} I_l^X\left(\widetilde{\left(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}\right) \otimes_0 \left(g_{l-q}^{(u,r)} \mathbf{1}_{(t,1)}^{\otimes l-q}\right)}\right)$$

where $g_0^{(u,r)} := b^{(u,r)}$ and $\sim$stands for symmetrization. Note that if $q = l$, then $l \leq m$ and

$$I_l^X\left(\widetilde{\left(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}\right) \otimes_0 \left(g_{l-q}^{(u,r)} \mathbf{1}_{(t,1)}^{\otimes l-q}\right)}\right) = b^{(u,r)} I_l^X\left(\left(h_l^{(u,r)} \mathbf{1}_{A_{l,l}(t)}\right)\right).$$

On the other hand, when $1 \leq q < l$, for every $\mathbf{x}_l \in [0,1]^l$

$$\begin{aligned}
\widetilde{\left(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}\right) \otimes_0 \left(g_{l-q}^{(u,r)} \mathbf{1}_{(t,1)}^{\otimes l-q}\right)} &= \binom{l}{q}^{-1} \sum_{\mathbf{j}_{(q)} \subset \{1,\ldots,l\}} h_q^{(u,r)}\left(\mathbf{x}_{\mathbf{j}_{(q)}}\right) g_{l-q}^{(u,r)}\left(\mathbf{x}_{\{1,\ldots,l\}\backslash \mathbf{j}_{(q)}}\right) \times \\
&\qquad \times \mathbf{1}_{[0,t)^q}\left(\mathbf{x}_{\mathbf{j}_{(q)}}\right) \mathbf{1}_{(t,1]^{l-q}}\left(\mathbf{x}_{\{1,\ldots,l\}\backslash \mathbf{j}_{(q)}}\right) \\
&= \binom{l}{q}^{-1} \mathbf{1}_{A_{l,q}(t)}\left(\mathbf{x}_l\right) \times \\
&\qquad \times \sum_{\mathbf{j}_{(q)} \subset \{1,\ldots,l\}} h_q^{(u,r)}\left(\mathbf{x}_{\mathbf{j}_{(q)}}\right) g_{l-q}^{(u,r)}\left(\mathbf{x}_{\{1,\ldots,l\}\backslash \mathbf{j}_{(q)}}\right) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}\left(\mathbf{x}_l\right).
\end{aligned}$$

Since the function

$$\mathbf{x}_l \mapsto \sum_{\mathbf{j}_{(q)} \subset \{1,\ldots,l\}} h_q^{(u,r)}\left(\mathbf{x}_{\mathbf{j}_{(q)}}\right) g_{l-q}^{(u,r)}\left(\mathbf{x}_{\{1,\ldots,l\}\backslash \mathbf{j}_{(q)}}\right) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}\left(\mathbf{x}_l\right)$$

is symmetric, we immediately deduce that, for every $r \geq 1$, the family of random variables

$$\left\{Z_t^{(r)} : t \in (0,1)\right\},$$

as defined in (20), admits a representation of the form (22), and namely

$$Z_t^{(r)} = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1 \leq q \leq l\wedge m} I_l^X\left(\mathbf{1}_{A_{l,q}(t)} f_{l,q}^{(r)}\right), \tag{23}$$

where

$$f_{l,q}^{(r)}(\mathbf{x}_l) := \binom{l}{q}^{-1} \sum_{u=1}^{M_r} \sum_{\mathbf{j}_{(q)} \subset \{1,\dots,l\}} h_q^{(u,r)}\left(\mathbf{x}_{\mathbf{j}_{(q)}}\right) g_{l-q}^{(u,r)}\left(\mathbf{x}_{\{1,\dots,l\}\setminus \mathbf{j}_{(q)}}\right) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}(\mathbf{x}_l).$$

Point (i) in the statement now follows from Theorem 6 and formula (23). Now suppose that

$$\lim_{r,r' \to +\infty} V\left(Z^{(r)} - Z^{(r')}\right) = 0.$$

Then, the existence of a process $Y$ satisfying (21) follows from the same arguments contained in the proof of Theorem 4. Moreover, relation (19) implies immediately that for every $l$ and $q$, the family $\left\{f_{l,q}^{(r)} : r \geq 1\right\}$ is a Cauchy sequence in $L_s^2\left([0,1]^l\right)$. Since $Y_t = L^2\text{-}\lim_{r\to+\infty} Z_t^{(r)}$ for every $t$, the conclusion is obtained by standard arguments. ■

Now, for every $p \geq 0$, call $\mathbf{BF}_p$ the subset of the class $\mathbf{BF}$, as defined through formula (7), composed of processes with the form (20) and such that $n + m \leq p$. We have therefore the following

**Proposition 8** *Fix $N \geq 0$, and consider a measurable process $Y = \{Y_t : t \in [0,1]\}$. Then, the following conditions are equivalent:*

1. *$Y \in \mathbf{FS}_N$;*
2. *there exists a sequence $Z^{(r)} \in \mathbf{BF}_{N+1}$, $r \geq 1$, such that $\lim_{r\to+\infty} V\left(Z^{(r)} - Y\right) = 0$*

**Proof.** The implication 2. $\Longrightarrow$ 1. is an immediate consequence of Lemma 7 and Theorem 6. To deal with the opposite direction, suppose that $Y_t = \delta\left(u\mathbf{1}_{[0,t]}\right)$, $t \in [0,1]$, where $u_\alpha(\omega) \in L^2([0,1]\times\Omega)$ is such that, for every $\alpha \in [0,1]$, $u_\alpha \in \oplus_{j=0,\dots,N} C_j$. Note that $u \in \mathbb{L}^{k,p}$ for every $k \geq 1$ and $p > 2$, and we can therefore take up the same line of reasoning and notation as in the proof of Theorem 1. In particular, according to Proposition 1 in Tudor (2004), we know that $Y$ admits the representation $Y_t = \int_0^t \mathbb{E}\left[v_\alpha \mid \mathcal{F}_{[\alpha,t]^c}\right] dX_\alpha$, where the process $v_\alpha = u_\alpha + \int_0^\alpha D_\alpha u_s dX_s$, $\alpha \in [0,1]$, is also such that $v_\alpha \in \oplus_{j=0,\dots,N} C_j$ for every $\alpha$. By linearity, this implies that for every partition $\pi = \{0 = t_0 < \dots < t_n = 1\}$ the random variables $F_i^\pi$, $i = 0,\dots,n-1$, as defined in (13), are such that $F_i^\pi \in \oplus_{j=0,\dots,N} C_j$. According to the remark following Lemma 2, every $F_i^\pi$ is the limit, say in $\mathbb{D}^{3,3}$, of a sequence of random variables with the form

$$G_m^{(i,m)} = \sum_{k=1}^{M_m} G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)}, \quad m \geq 1,$$

where $M_m \geq 1$ for every $m$, and also

$$\begin{aligned} G_{m,k}^{(i,\pi,1)} &= a + \sum_{l=1}^{\gamma_1} I_l^X\left(h_l \mathbf{1}_{(0,t_j)^l}\right) \\ G_{m,k}^{(i,\pi,2)} &= b + \sum_{r=1}^{\gamma_2} I_r^X\left(g_r \mathbf{1}_{(t_{j+1},1)^r}\right) \end{aligned}$$

where all dependencies on $i, \pi, m$ and $k$ have been dropped in the second members, and $\gamma_1 + \gamma_2 \leq N+1$. By using relations (16) and (17), we see immediately that the process $U_t^{(m,k,\pi,i)}$, $t \in [0,1]$, is an element of $\mathbf{BF}_{N+1}$, and the conclusion is obtained as in the proof of Theorem 1. ■

## 5 Skorohod integrals as time-reversed Brownian martingales

Now fix $k \geq 3$ and $p > 2$, take $u \in \mathbb{L}^{k,p}$, and note $Y_t = \delta(u\mathbf{1}_{[0,t]})$. Suppose moreover that the process $v_\alpha \in \mathbb{L}^{k-2,p}$ appearing in formula (10) is such that $v_\alpha = D_\alpha F$ for some $F \in \mathbb{D}^{1,2}$ (we refer to Nualart (1995, p. 40) for a characterization of such processes in term of their Wiener-Itô expansion). Then, according to the *generalized Clark-Ocone formula* stated in Nualart and Pardoux (1988),

$$Y_t = \int_0^t \mathbb{E}\left[D_\alpha F \mid \mathcal{F}_{[\alpha,t]^c}\right] dX_\alpha = F - \mathbb{E}\left[F \mid \mathcal{F}_{t^c}\right], \quad t \in [0,1]. \tag{24}$$

As made clear by the following discussion, a process of the type $Y_t = F - \mathbb{E}[F \mid \mathcal{F}_{t^c}]$ can be easily represented as a *time-reversed Brownian martingale*. The principal aim of this section is to establish sufficient conditions to have that $Y_t$ is a semimartingale in its own filtration (the reader is referred to Tudor (2004), for further applications of (24) to Skorohod integration).

To this end, for every $f \in L^2([0,1])$ we define $\widehat{f}(x) = f(1-x)$, so that the transformation $f \mapsto \widehat{f}$ is an isomorphism of $L^2([0,1])$ into itself. Such an operator can be extended to the space $L_s^2([0,1]^n)$ – i.e. the space of square integrable and symmetric functions on $[0,1]^n$ – by setting

$$\widehat{f}_n(x_1,...x_n) = f(1-x_1,...,1-x_n)$$

for every $f_n \in L_s^2([0,1]^n)$, thus obtaining an isomorphism of $L_s^2([0,1]^n)$ into itself. We also set, for $f \in L^2([0,1])$, $\widehat{X}(f) = X\left(\widehat{f}\right)$ and eventually

$$\widehat{X} = \left\{\widehat{X}(f) : f \in L^2([0,1])\right\}.$$

Of course, $\widehat{X}$ is an isonormal Gaussian process on $L^2([0,1])$, and the random function

$$\widehat{X}_t = \widehat{X}\left(\mathbf{1}_{[0,t]}\right) = X_1 - X_{1-t}, \quad t \in [0,1],$$

is again a standard Brownian motion. As usual, given $n \geq 1$ and $h_n \in L_s^2([0,1]^n)$, $I_n^X(h_n)$ and $I_n^{\widehat{X}}(h_n)$ stand for the multiple Wiener-Itô integrals of $h_n$, respectively with respect to $X$ and $\widehat{X}$ (see Nualart (1995)). The following lemma will be useful throughout the sequel.

**Lemma 9** *Let $F \in L^2(\mathbb{P})$ have the Wiener-Itô expansion $F = \mathbb{E}(F) + \sum_{n=1}^\infty I_n^X(f_n)$, then*

$$F = \mathbb{E}(F) + \sum_{n=1}^\infty I_n^{\widehat{X}}\left(\widehat{f}_n\right).$$

**Proof.** By density, one can consider functionals with the form $F = I_n^X(f^{\otimes n})$, $n \geq 1$, where $f \in L^2([0,1])$ and $f^{\otimes n}(x_1,...x_n) = f(x_1)...f(x_n)$. In this case, it is well known that $F = n!H_n(X(f))$, where $H_n$ is the $n$th Hermite polynomial as defined in Nualart (1990, Ch. 1), and therefore

$$F = n!H_n\left(\widehat{X}\left(\widehat{f}\right)\right) = I_n^{\widehat{X}}\left(\widehat{f}^{\otimes n}\right) = I_n^{\widehat{X}}\left(\widehat{f^{\otimes n}}\right),$$

thus proving the claim. ■

We now introduce the following filtration:

$$\widehat{\mathcal{F}}_t = \sigma\left\{\widehat{X}_h : h \leq t\right\}, \quad t \in [0,1].$$

Note that

$$\begin{aligned} \mathcal{F}_{[s,t]^c} &= \mathcal{F}_s \vee \widehat{\mathcal{F}}_{1-t} \\ \mathcal{F}_{t^c} &= \widehat{\mathcal{F}}_{1-t}. \end{aligned} \tag{25}$$

**Proposition 10** *Let $\{Y_t : t \in [0,1]\}$ be a measurable process.*

1. *The following conditions are equivalent,*

**(i)** *there exists $F \in L^2(\mathbb{P})$ such that $Y_t = F - \mathbb{E}(F \mid \mathcal{F}_{t^c})$;*

**(ii)** *there exists a square integrable $\widehat{\mathcal{F}}_t$ - martingale $\left\{\widehat{M_t} : t \in [0,1]\right\}$ such that $Y_t = \widehat{M}_1 - \widehat{M}_{1-t}$;*

**(iii)** *there exists a $\widehat{\mathcal{F}}_\alpha$ - predictable process $\left\{\widehat{\phi}_\alpha : \alpha \in [0,1]\right\}$, such that $\mathbb{E}\left(\int_0^1 \widehat{\phi}_\alpha^2 d\alpha\right) < +\infty$ and $Y_t = \int_{1-t}^1 \widehat{\phi}_\alpha d\widehat{X}_\alpha$;*

**(iv)** *there exist kernels $f_n \in L_s^2([0,1]^n)$, $n \geq 1$, such that*

$$Y_t = \sum_{n=1}^{\infty} I_n^X\left(f_n\left(1 - \mathbf{1}_{[t,1]}^{\otimes n}\right)\right) = \sum_{n=1}^{\infty} I_n^{\widehat{X}}\left(\widehat{f}_n\left(1 - \mathbf{1}_{[0,1-t]}^{\otimes n}\right)\right),$$

*where the convergence of the series takes place in $L^2(\mathbb{P})$.*

2. *Let either one of conditions (i)-(iv) be verified, and let $F$ be given by (i) and the $f_n$'s by (iv). Then,*

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^X(f_n) = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^{\widehat{X}}\left(\widehat{f}_n\right).$$

3. *Under the assumptions of point 2, suppose moreover that $F$ is an element of $\mathbb{D}^{1,2}$, and let $\widehat{\phi}$ be given by (iii). Then,*

$$\widehat{\phi}_\alpha = \mathbb{E}\left[D_{1-\alpha}F(X) \mid \widehat{\mathcal{F}}_\alpha\right], \quad \alpha \in [0,1], \tag{26}$$

*where $DF(X)$ is the usual Malliavin derivative of $F$, regarded as a functional of $X$.*

**Remark** – Note that formula (26) above appears also in Wu (1990, formula (4.4)), where it is obtained by completely different arguments.

**Proof.** If (i) is verified, then (ii) holds, thanks to (25), by defining $\widehat{M_t} = \mathbb{E}\left(F \mid \widehat{\mathcal{F}}_t\right)$. On the other hand, (ii) implies (iii) due to the predictable representation property of $\widehat{X}$. Of course, if (iii) is verified, then

$$Y_t = \int_{1-t}^1 \widehat{\phi}_\alpha d\widehat{X}_\alpha = \int_0^1 \widehat{\phi}_\alpha d\widehat{X}_\alpha - \int_0^{1-t} \widehat{\phi}_\alpha d\widehat{X}_\alpha = F - \mathbb{E}\left[F \mid \widehat{\mathcal{F}}_{1-t}\right],$$

where $F = \int_0^1 \widehat{\phi}_\alpha d\widehat{X}_\alpha$, thus proving the implication (iii) $\Longrightarrow$ (i). Now, let (i) be verified, and let $F$ have the representation

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^X(f_n);$$

we may apply Lemma 1.2.4 in Nualart (1995) to obtain that

$$Y_t = \sum_{n=1}^{\infty} I_n^X(f_n) - \mathbb{E}\left[\sum_{n=1}^{\infty} I_n^X(f_n) \mid \mathcal{F}_{t^c}\right] = \sum_{n=1}^{\infty} I_n^X(f_n) - \sum_{n=1}^{\infty} I_n^X\left(f_n \mathbf{1}_{[t,1]}^{\otimes n}\right) \tag{27}$$

thus giving immediately (i) $\Longrightarrow$ (iv) (the second equality in (iv) is a consequence of Lemma 9). The opposite implication may be obtained by reading backwards formula (27). The proof of point 2 is now immediate. To deal with point 3, observe if $F$ is derivable in the Malliavin sense as a functional of $X$, then $F$ is also derivable as a functional of $\widehat{X}$, and the two derivative processes must verify

$$D_\alpha F\left(\widehat{X}\right) = D_{1-\alpha}F(X), \quad \text{a.e.} - d\alpha \otimes d\mathbb{P},$$

where $DF\left(\widehat{X}\right)$ stands for the Malliavin derivative of $F$, regarded as a functional of $\widehat{X}$. As a matter of fact, let $F_k$ be a sequence of polynomial functionals with the form $F_k = p\left(X\left(h_1\right), ..., X\left(h_m\right)\right)$, where $p$ is a polynomial in $m$ variables (note that $p$, $m$ and the $h_j$'s may in general depend on $k$), converging to $F$ in $L^2\left(\mathbb{P}\right)$ and satisfying

$$\mathbb{E}\left[\int_0^1 \left(\sum_{j=1}^m \frac{\partial}{\partial x_j} p\left(X\left(h_1\right), ..., X\left(h_m\right)\right) h_j\left(x\right) - D_x F\left(X\right)\right)^2 dx\right] \to 0.$$

Then, $p\left(X\left(h_1\right), ..., X\left(h_m\right)\right) = p\left(\widehat{X}\left(\widehat{h}_1\right), ..., \widehat{X}\left(\widehat{h}_m\right)\right)$, and also

$$\mathbb{E}\left[\int_0^1 \left(\sum_{j=1}^m \frac{\partial}{\partial x_j} p\left(\widehat{X}\left(\widehat{h}_1\right), ..., \widehat{X}\left(\widehat{h}_m\right)\right) \widehat{h}_j\left(x\right) - D_{1-x} F\left(X\right)\right)^2 dx\right] \to 0,$$

thus giving immediately the desired conclusion. The proof of point 3 is achieved by using the Clark-Ocone formula (see Clark (1970) and Ocone (1984)). ■

**Example** – Let $F = H_n\left(X\left(h\right)\right)$, where $H_n$ is the $n$th Hermite polynomial and $h$ is such that $\|h\| = 1$. Then, thanks to Proposition 10-2 the process $Y_t = F - \mathbb{E}\left[F \mid \mathcal{F}_{t^c}\right]$ has the representation

$$\begin{aligned} Y_t &= \frac{1}{n!}\left[I_n^X\left(h^{\otimes n}\right) - I_n^X\left(h^{\otimes n}\mathbf{1}_{[t,1]}^{\otimes n}\right)\right] = H_n\left(X\left(h\right)\right) - \left\|h\mathbf{1}_{[0,t]}\right\|^n I_n^X\left(\left(\frac{h\mathbf{1}_{[t,1]}}{\left\|h\mathbf{1}_{[t,1]}\right\|}\right)^{\otimes n}\right) \qquad (28) \\ &= H_n\left(X\left(h\right)\right) - \left\|h\mathbf{1}_{[t,1]}\right\|^n H_n\left(\frac{X\left(h\mathbf{1}_{[t,1]}\right)}{\left\|h\mathbf{1}_{[t,1]}\right\|}\right) \end{aligned}$$

as well as

$$Y_t = \int_{1-t}^1 \mathbb{E}\left[H_{n-1}\left(X\left(h\right)\right) \mid \widehat{\mathcal{F}}_\alpha\right] h\left(1-\alpha\right) d\widehat{X}_\alpha.$$

Formula (28) generalizes the obvious relations (corresponding to the case $n = 1$ and $h = \mathbf{1}_{[0,1]}$)

$$X_1 - \mathbb{E}\left[X_1 \mid \mathcal{F}_{t^c}\right] = X_t = \widehat{X}_1 - \widehat{X}_{1-t}$$

Given a filtration $\{\mathcal{G}_t : t \in [0,1]\}$, and two adapted, cadlag processes $U_t$, and $V_t$, we will write $[U, V] = \{[U, V]_t : t \in [0,1]\}$ to indicate the *quadratic covariation process* of $U$ and $V$ (if it exists). This means that $[U, V]$ is the cadlag $\mathcal{G}_t$ - adapted process of bounded variation such that, for every $t \in [0,1]$ and for every sequence of (possibly random) partitions of $[0, t]$ – say $\tau_n = \{0 < t_{1,n} < ... < t_{M_n,n} = t\}$ – with mesh tending to zero, the sequence

$$\lim_n \left[U_0 V_0 + \sum_{i=0}^{M_n - 1}\left(U_{t_{i+1,n}} - U_{t_{i,n}}\right)\left(V_{t_{i+1,n}} - V_{t_{i,n}}\right)\right] = [U, V]_t$$

where the convergence is in probability, and uniform on compacts. The next result uses quadratic covariations to characterize processes of the form $t \mapsto \left(F - \mathbb{E}\left[F \mid \mathcal{F}_{t^c}\right]\right)$ in terms of semimartingales.

**Proposition 11** *Let $F$ and $\{Y_t : t \in [0,1]\}$ satisfy either one of conditions (i)-(iv) in Proposition 10, fix $k \geq 1$, and let $\widehat{\phi}_\alpha$, as in Proposition 10-1-(iii), be càdlàg and of the form*

$$\widehat{\phi}_\alpha = \Phi\left(\alpha; \widehat{X}\left(g_1 \mathbf{1}_{[0,\alpha]}\right), ..., \widehat{X}\left(g_k \mathbf{1}_{[0,\alpha]}\right)\right)$$

*where $\Phi$ is a measurable function on $[0,1] \times \Re^k$, and $g_j \in L^2([0,1])$, $j = 1, ..., k$. If there exists the quadratic covariation process $\left[\widehat{\phi}, \widehat{X}\right]$, then $Y_t$ is a semimartingale on $[0,1]$ in its own filtration, and moreover*

$$Y_t = \int_0^t \widehat{\phi}_{1-\alpha} dX_\alpha - \left[\widehat{\phi}, \widehat{X}\right]_1 + \left[\widehat{\phi}, \widehat{X}\right]_{1-t}. \tag{29}$$

**Proof.** The proof is directly inspired by Theorem 3.3 in Jacod and Protter (1988). Let $t \in (0,1]$ and $\tau = \{1 - t = s_0 < ... < s_n = 1\}$ be a deterministic partition of $[1-t, 1]$. Then, when the mesh of $\tau$ converges to zero, $Y_t$ is (uniformly) the limit in probability of

$$\sum_{i=0}^{n-1} \widehat{\phi}_{s_i} \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right).$$

Now note that, since $\widehat{X}\left(g_j \mathbf{1}_{[0,1-\alpha]}\right) = X\left(\widehat{g}_j\right) - X\left(\widehat{g}_j \mathbf{1}_{[0,\alpha]}\right)$, $j = 1, ..., k$, the process $\alpha \mapsto \widehat{\phi}_{1-\alpha}$ is left-continuous and adapted to the filtration

$$\mathcal{H}_\alpha = \sigma\left(X_h,\ h \leq \alpha\right) \vee \sigma\left(X_1, X\left(\widehat{g}_1\right), ..., X\left(\widehat{g}_k\right)\right), \quad \alpha \in [0,1].$$

Therefore, since $X_t$ is classically a $\mathcal{H}_t$ - semimartingale (see Chaleyat-Mauriel and Jeulin (1983)), the stochastic integral in (29) is well defined as the limit in probability of the sequence

$$\sum_{i=0}^{n-1} \widehat{\phi}_{1-t_{i+1}} \left(X_{t_i} - X_{t_{i+1}}\right) = \sum_{i=0}^{n-1} \widehat{\phi}_{s_{i+1}} \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right)$$

where $t_i = 1 - s_i$. Eventually, we shall observe that the finite variation process $t \mapsto \left[\widehat{\phi}, \widehat{X}\right]_1 - \left[\widehat{\phi}, \widehat{X}\right]_{1-t}$ is by definition the limit in probability (as the mesh of $\tau$ converges to zero) of

$$\sum_{i=0}^{n-1} \left(\widehat{\phi}_{s_{i+1}} - \widehat{\phi}_{s_i}\right) \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right),$$

and therefore it is a $\mathcal{H}_t$ - semimartingale, being an adapted process of finite variation (to prove the adaptation, just observe that if $1 - t \leq s \leq 1$, then

$$\begin{aligned} \widehat{\phi}_s &= \Phi\left(\alpha; X\left(\widehat{g}_1\right) - X\left(\widehat{g}_1 \mathbf{1}_{[0,1-s]}\right), ..., X\left(\widehat{g}_k\right) - X\left(\widehat{g}_k \mathbf{1}_{[0,1-s]}\right)\right) \\ &\in \sigma\left(X_h,\ h \leq t\right) \vee \sigma\left(X_1, X\left(\widehat{g}_1\right), ..., X\left(\widehat{g}_k\right)\right)). \end{aligned}$$

As a consequence of the above discussion, the quantity

$$Y_t - \int_0^t \widehat{\phi}_{1-\alpha} dX_\alpha + \left[\phi, \widehat{X}\right]_1 - \left[\phi, \widehat{X}\right]_{1-t}$$

is the limit in probability of

$$\sum_{i=0}^{n-1} \widehat{\phi}_{s_i} \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right) - \sum_{i=0}^{n-1} \widehat{\phi}_{s_{i+1}} \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right) + \sum_{i=0}^{n-1} \left(\widehat{\phi}_{s_{i+1}} - \widehat{\phi}_{s_i}\right) \left(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\right)$$

which equals zero for every $\tau$. To conclude, observe that $Y_t$ is the sum of two $\mathcal{H}_t$ - semimartingales, and therefore it is itself a $\mathcal{H}_t$ - semimartingale and consequently, by Stricker's theorem, it is a semimartingale in its own filtration. ■

Now we state a (classic) sufficient condition for the existence of the quadratic covariation process $\left[\widehat{\phi}, \widehat{X}\right]$.

**Proposition 12** *Under the assumptions and notation of Proposition 11, suppose that the function $\Phi$ is of class $C^1$ in $[0,1]\times\Re^k$. Then, the quadratic covariation process $\left[\widehat{\phi},\widehat{X}\right]$ exists.*

**Proof.** This is an application of Theorem 5 in Meyer (1976, p. 359). The vector

$$\gamma_\alpha := \left(\alpha, \widehat{X}_\alpha, \widehat{X}\left(g_1\mathbf{1}_{[0,\alpha]}\right), ..., \widehat{X}\left(g_k\mathbf{1}_{[0,\alpha]}\right)\right)$$

is indeed a $(k+2)$ - dimensional $\widehat{\mathcal{F}}_\alpha$ - semimartingale. Now define

$$\Phi^*\left(\alpha, x_1, ..., x_{k+1}\right) = \Phi\left(\alpha, x_2, ..., x_{k+1}\right), \quad \left(\alpha, x_1, ..., x_{k+1}\right) \in [0,1]\times\Re^{k+1}.$$

Since the assumptions imply that $\Phi^*$ is of class $C^1$ in $[0,1]\times\Re^{k+1}$ and $\widehat{\phi}_\alpha = \Phi^*\left(\gamma_\alpha\right)$, the quadratic variation process $\alpha \mapsto \left[\widehat{\phi},\widehat{\phi}\right]_\alpha$ exists, as well as the processes $\left[\widehat{X},\widehat{X}\right]$ and $\left[\widehat{\phi}+\widehat{X},\widehat{\phi}+\widehat{X}\right]$. It follows that $\left[\widehat{\phi},\widehat{X}\right]$ exists, thanks to the polarization identity

$$\left[\widehat{\phi},\widehat{X}\right]_\alpha = \frac{1}{2}\left\{\left[\widehat{\phi}+\widehat{X},\widehat{\phi}+\widehat{X}\right]_\alpha - \left[\widehat{X},\widehat{X}\right]_\alpha - \left[\widehat{\phi},\widehat{\phi}\right]_\alpha\right\}, \quad \alpha \in [0,1].$$

■

# 6 Anticipating integrals and stopping times

For the sake of completeness, in this section we explore some links between Skorohod integral processes and the family of stopping times. Classically, the stopping times are strongly related to the martingale theory. For instance, fix a filtration $\mathcal{U}_t$ as well as a $\mathcal{U}_t$ - stopping time $T$: it is well known, from the *Optional Sampling Theorem* (see e.g. Chung (1974)), that, for any $\mathcal{U}_t$ - martingale $M_t$, the stopped process $t \mapsto M_{T\wedge t}$ is again a martingale for the filtration $t \mapsto \mathcal{U}_{T\wedge t}$ of events determined prior to $T$. It is also well-known that, a stopped Itô integral at the stopping time $T$ coincides with the Itô integral on the random interval $[0,T]$. In this section, we prove a variant of the Optional Sampling Theorem for Skorohod integral processes and we discuss what happens if one samples such a process at a random time. For a discussion in this direction, see also the paper Nualart and Thieullen (1994). We keep the notation of the previous sections, and consider anticipating integral processes given by

$$Y_t = \delta\left(u\mathbf{1}_{[0,t]}(\cdot)\right)$$

where $u\mathbf{1}_{[0,t]}$ belongs to $Dom(\delta)$ for every $t\in[0,1]$. Given two stopping times $S$, $T$ for the filtration $\mathcal{F}_t$, we denote by $\mathcal{F}_T$, resp. $\mathcal{F}_S$, the $\sigma$-field of the events determined prior to $T$, resp. $S$.

We have the following *Optional Sampling Theorem.*

**Proposition 13** *If $S,T$ are $\mathcal{F}_t$- stopping times such that $S\le T$ a.s., it holds that*

$$E\left[Y_T - Y_S|\mathcal{F}_S\right] = 0. \tag{30}$$

**Proof.** Let us first consider as in Karatzas and Shreve (1991) two sequences of stopping times $(S_n)_n, (T_n)_n$ taking on a countable number of values in the dyadic partition of $[0,1]$ and such that $S_n \to S$, $T_n \to T$ and

$$S \le S_n\ ,\ T\le T_n \text{ and } S_n \le T_n.$$

As in Chung (1974), p. 325, using the fact that the process $\left(\mathbb{E}\left(Y_t|\mathcal{F}_t\right)\right)_t$ is a martingale, we can prove that $\int_A Y_{S_n} d\mathbb{P} = \int_A Y_{T_n} d\mathbb{P}$ for every $A\in\mathcal{F}_{S_n}$. We follow next the lines of the proof of Theorem 1.3.22 in Karatzas and Shreve (1991), observing that the sequence $(Y_{S_n})_n$ is uniformly integrable. This is consequence of the bound

$$\sup_t \mathbb{E}Y_t^2 \le \sup_t\left(\mathbb{E}(Y_1-Y_t)^2 + \mathbb{E}Y_t^2\right) \le V(Y).$$

■

The next result is a version of Theorem 2.5 of Nualart and Thieullen (1994).

**Proposition 14** *Let $u \in \mathbb{L}^{1,p}$, $p > 4$, and let $T$ be a stopping time for the filtration $\mathcal{F}_t$. Then $u\mathbf{1}_{[0,T]}$ belongs to $Dom(\delta)$ and it holds*

$$\delta(u\mathbf{1}_{[0,t]}) \mid_{t=T} = \delta(u\mathbf{1}_{[0,T]}). \tag{31}$$

**Proof.** Since, for $u$ as in the statement, the process $t \mapsto \int_0^t \mathbb{E}(u_s)\, dX_s$ is a continuous, square integrable Gaussian $\mathcal{F}_t$ - martingale, we can assume, without loss of generality, that $\mathbb{E}(u_t) = 0$ for every $t \in [0,1]$. We first prove the property (31) for the approximation $u^\pi$ given by (11)

$$u_t^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left( \int_{t_i}^{t_{i+1}} E\left(u_s \mid \mathcal{F}_{[t_i,t_{i+1}]^c}\right) ds \right) \mathbf{1}_{[t_i,t_{i+1}]}(t).$$

Let us consider the sum

$$S = \sum_{i=0}^{n-1} F_i \left( X_{T\wedge t_{i+1}} - X_{T\wedge t_i} \right) = \sum_{i=0}^{n-1} F_i \delta(\mathbf{1}_{[0,T]}\mathbf{1}_{[t_i,t_{i+1}]})$$

where $F_i = \frac{1}{t_{i+1}-t_i} \left( \int_{t_i}^{t_{i+1}} E\left(u_s \mid \mathcal{F}_{[t_i,t_{i+1}]^c}\right) ds \right)$. Using relation (6) (note that all hypothesis are satisfied, that is, $F_i \in \mathbb{D}^{1,2}$, $\mathbf{1}_{[0,T]}\mathbf{1}_{[t_i,t_{i+1}]} \in Dom(\delta)$, being adapted, and $\mathbb{E}\left( F^2 \int_0^1 \mathbf{1}_{[0,T]}(s)\mathbf{1}_{[t_i,t_{i+1}]}(s) ds \right) \leq \mathbb{E}(F^2) < \infty$ ) and (5), we obtain that $u^\pi \mathbf{1}_{[0,T]} \in Dom(\delta)$ and

$$\delta(u^\pi \mathbf{1}_{[0,T]}) = S = \sum_{i=0}^{n-1} F_i \left( X_{t\wedge t_{i+1}} - X_{t\wedge t_i} \right) \mid_{t=T} = \delta(u^\pi \mathbf{1}_{[0,t]}) \mid_{t=T}.$$

Now recall that, for every partition $\pi$, the process $u^\pi$ is an element of $\mathbb{L}^{1,p}$, and also, when $|\pi| \to 0$,

$$\begin{array}{ll} u^\pi \to u & \text{in } \mathbb{L}^{1,p} \\ u^\pi \mathbf{1}_{[0,T]} \to u\mathbf{1}_{[0,T]} & \text{in } L^2([0,1]\times\Omega) \\ \delta\left(u^\pi \mathbf{1}_{[0,t]}\right) \to \delta\left(u\mathbf{1}_{[0,t]}\right) & \text{in } L^2(\mathbb{P}) \text{ for every } t \in [0,1]. \end{array} \tag{32}$$

Fix a sequence of partitions $\pi$ such that $|\pi| \to 0$. From (32), we deduce immediately that there exists a finite constant $K > 0$, not depending on $\pi$, such that

$$\int_0^1 \mathbb{E}\left[ \left| \int_0^1 \left(D_s u_t^\pi\right)^2 ds \right|^{\frac{p}{2}} \right] dt < K, \quad \text{for every } \pi.$$

Moreover, since $\mathbb{E}(u_t^\pi) = 0$ for every $t$, we can use the same line of reasoning as in the proof of Nualart (1998, Proposition 5.1.1), and deduce the existence of a finite constant $K' > 0$ such that, for every $s,t \in [0,1]$ and every $\pi$,

$$\mathbb{E}\left[ \left| \delta\left(u^\pi \mathbf{1}_{[0,t]}\right) - \delta\left(u^\pi \mathbf{1}_{[0,s]}\right) \right|^p \right] \leq K' \times |t-s|^{\frac{p}{2}-1}.$$

As a consequence, by applying for instance Nualart (1998, Lemma 5.3.1), and since $T$ takes values in $[0,1]$ by construction, we deduce that, as $|\pi| \to 0$,

$$\delta\left(u^\pi \mathbf{1}_{[0,T]}\right) = \delta\left(u^\pi \mathbf{1}_{[0,t]}\right)\Big|_{t=T} \to \delta\left(u\mathbf{1}_{[0,t]}\right)\Big|_{t=T} \quad \text{in } L^p(\mathbb{P}).$$

We conclude by the basic lemma for the convergence of Skorohod integrals that $u\mathbf{1}_{[0,T]} \in Dom(\delta)$ and (31) holds. ■

**Remark** – Note that in Nualart and Thieullen (1994, Theorem 2.5) the authors proved the following relation, for every $\mathcal{F}_t$-stopping time $T$ and for every $u \in Dom(\delta)$,

$$\delta(u\mathbf{1}_{[0,T]}) = \delta(u\mathbf{1}_{[0,t]}) \mid_{t=T^+}$$

where $\delta(u\mathbf{1}_{[0,t]}) \mid_{t=T^+}$ is defined as

$$\delta(u\mathbf{1}_{[0,t]}) \mid_{t=T^+} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_T^{T+\varepsilon} \delta(u\mathbf{1}_{[0,s]}) ds$$

when the above limit exists in $L^2(\mathbb{P})$. The obtention of the result (31) is due to the use of the approximating processes (11) for which the limit can be explicitly computed. Note that, with our method, we do not need to introduce any special assumption on $T$. On the other hand, we are forced to assume a stronger hypothesis on the integrand $u$, that is, $u \in \mathbb{L}^{1,p}$, $p > 4$, instead of $u \in Dom(\delta)$.

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